# M.Sc. II Sem. (Mathematics)

# Paper 1st - Advanced Abstract Algebra-II

# Unit - V

**Reference Book :**

- **P.B.Bhattacharya, S.K. Jain and S.R Nagpaul, *Basic Abstract Algebra* (2nd Edition), Cambridge University Press, Indian Edition, 1997.**
- **C. Musili, *Introduction to Rings and Modules,* Second Revised Edition, Narosa Publishing House, New Delhi.**
- **I.N. Herstein, Topics in Algebra, Wiley Easter Ltd., New Delhi, 1975.**

**Definition.** A non-empty set V is said to be a vector space over a field F if V is an abelian group with respect to addition and if for every $\alpha \in F$, $v \in V$, there exists $\alpha v \in V$ such that

1. $\alpha(v + w) = \alpha v + \alpha w$ ;
2. $(\alpha + \beta)v = \alpha v + \alpha v$ ;
3. $\alpha(\beta v) = (\alpha\beta)v$ ;
4. $1v = v$

for all $\alpha, \beta \in F$, $v, w \in V$ and $1 \in F$.

We shall consistently use the following notations :

a) F will be a field.

b) Lowercase Greek letters are the elements of F and we shall refer to the elements of F as scalars.

c) Capital letters will denote vector spaces over F.

d) Lowercase Latin letters will denote elements of vector spaces and we shall refer to the elements of a vector space as vectors.

**Definition.** A commutative division ring is called a field.

**Definition.** A field is a commutative ring with unit element in which every non-zero element has a multiplicative inverse.

**Remark.** Let V be a vector space over a field F and suppose Hom (V, W) be the set of all vector space homomorphisms of V into W. Then Hom (V, W) forms a vector space over a field F with the operations ( + ) and ( . ) as follows :

1. $(T_1 + T_2)(x) = T_1(x) + T_2(x)$, where $T_1, T_2 \in$ Hom (V, W) and $x \in V$;
2. $(\alpha.T)(x) = \alpha.T(x)$, where $T \in$ Hom (V, W), $x \in V$ and $\alpha \in F$.

**Definition.** An associative ring A is called an algebra over F if A is a vector space over F such that for all $a, b \in A$ and $\alpha \in F$,

$$\alpha(ab) = (\alpha a)b = a(\alpha b).$$

For example, Hom (V, V) is an algebra over F.

For convenience, we write Hom (V, V) as A(V). If want to emphasize the importance of field we shall denote it by $A_F(V)$.

**Definition.** A linear transformation on V, over F, is an element of $A_F(V)$.

**Definition.** Let V be an n-dimensional vector space over a field F. The linear transformations $S, T \in A(V)$ are said to be similar if there exists an invertible element $C \in A(V)$ such that

$$T = CSC^{-1}.$$

**Definition.** The subspace W of V is invariant under $T \in A(V)$ if $WT \subset W$.

**Definition.** An element $T \in A(V)$ is called right invertible if there exists an element $S \in A(V)$ such that $TS = 1$. [Here 1 denotes the unit element of A(V)].

Similarly, if there is a $U \in A(V)$ such that $UT = 1$, then $T \in A(V)$ is called left invertible.

**Definition.** An element T in A(V) is invertible or regular if it is both right and left invertible, i.e. if there is element $S \in A(V)$ such that $ST = TS = 1$.

We write S as $T^{-1}$.

**Definition.** An element in A(V) which is not regular is singular.

It is quite possible that an element in A(V) is right invertible but is not invertible. For example, suppose F be the field of real numbers and suppose V be F[x], the set of all polynomials in x over F.

In V, suppose S be defined by

$$q(x)S = \frac{d}{dx}q(x)$$

and T by

$$q(x)T = \int_1^x q(x)dx.$$

Then $ST \neq 1$, whereas $TS = 1$.

**Remark.** If V is finite dimensional over F, then an element in A(V) which is right invertible is invertible.