Normal subgroup: A subgroup N of a group G is normal if $Nx = xN$ for all $x \in G$.
or $xNx^{-1} = N$
If N is a normal subgroup of G, we write $N \triangleleft G$.

Quotient (or factor) group: If N is a normal subgroup of a group G, then the group G/N is called the quotient group of G by N.

Remark: $\mathrm{Ker}(\phi)$ and $Z(G)$ is a normal subgroup of G.

Fundamental theorem of homomorphism (first isomorphism theorem):
Every homomorphic image of G is isomorphic to a quotient group of G.
Let $\phi : G \to G'$ be a homomorphism of groups. Then

$$G/\mathrm{Ker}(\phi) \simeq \mathrm{Im}(\phi).$$

Theorem. Let G be a group, then $\mathrm{In}(G)$ is a subgroup of $\mathrm{Aut}(G)$ and $G/Z(G) \simeq \mathrm{In}(G)$.

Proof. Define $\phi : G \to \mathrm{Aut}(G)$ by

$$\phi(a) = I_a$$

Clearly, ϕ is well-defined.
For any $a, b \in G$, we have

$$\begin{aligned}
\phi(ab)(x) &= I_{ab}(x) \quad , \forall x \in G \\
&= ab(x)(ab)^{-1} \quad \text{(by definition of } I_a(x)) \\
&= abxb^{-1}a^{-1} \quad (\because (ab)^{-1} = b^{-1}a^{-1}) \\
&= a(bxb^{-1})a^{-1} \\
&= I_a I_b(x) \quad , \forall x \in G.
\end{aligned}$$

i.e. $\phi(ab) = \phi(a).\phi(b)$.

$\Rightarrow$ ϕ is a homomorphism, and therefore,

$In(G) = Im(\phi)$ is a subgroup of $Aut(G)$.

Further, I_a is the identity automorphism if and only if $axa^{-1} = x \quad \forall x \in G$

$= \{x \mid \phi(a) = I_a\}$.

Now, $Ker\,\phi = \{a \in G \mid I_a(x) = x, \forall x \in G\}$

$= \{a \in G \mid axa^{-1} = x, \forall x \in G\}$

$= \{a \in G \mid ax = xa, \forall x \in G\}$

$= Z(G)$, centre of a group G, is a normal subgroup of G.

i.e. $Ker(\phi)$ is a normal subgroup of G.

And so by the fundamental theorem of homomorphism,

$$G/Z(G) \simeq In(G).$$

Hence proved.

Theorem: Show that $In(G) \lhd Aut(G)$.

Proof. For any $\sigma \in Aut(G)$ and $x \in G$, we have

$$(\sigma I_a \sigma^{-1})(x) = (\sigma(axa^{-1})\sigma^{-1})$$
$$= \sigma(a)x(a^{-1}\sigma^{-1})$$
$$= \sigma(a)x(\sigma(a))^{-1}$$
$$= I_{\sigma(a)}(x)$$

i.e. $\sigma I_a \sigma^{-1} = I_{\sigma(a)} \in In(G)$. [$\because N \lhd G$ iff $xNx^{-1} = N \; \forall x \in G$]

$\therefore$ $In(G) \lhd Aut(G)$.

Note: The group $Aut(G)/In(G)$ is called the group of outer automorphisms of G.