M.Sc. II Sem. (Mathematics)

Paper 1st - Advanced Abstract Algebra-II

Unit - IV

Reference Book :

- **P.B.Bhattacharya, S.K. Jain and S.R Nagpaul, *Basic Abstract Algebra* (2nd Edition), Cambridge University Press, Indian Edition, 1997.**
- **C. Musili, *Introduction to Rings and Modules,* Second Revised Edition, Narosa Publishing House, New Delhi.**

Topic : Uniform Modules, Primary Modules and Noether-Lasker Theorem

Definition 1. A non-zero module M is called **uniform** if any two non-zero submodules of M have non-zero intersection.

Definition 2. Two uniform modules U and V are said to be **subisomorphic**, denoted by **U ~ V** if U and V contain non-zero isomorphic submodules.

Here ~ is an equivalence relation and [U] denotes the equivalence class of U.

Definition 3. A module M is called **primary** if each non-zero submodule of M has uniform submodule and any two uniform submodules of M are subisomorphic.

Example. Z as a Z-module is uniform and primary. Indeed, any uniform module must be primary. Another example of a uniform module is a commutative integral domain regarded as a module over itself.

Theorem 1. Let M be a Noetherian module or a module over a Noetherian ring. Then each non-zero submodule of M contains a uniform module.

Proof. Let $0 \neq X \in M$. It is enough to show Rx contains a uniform submodule. If M is noetherian, the submodule Rx is also noetherian. But if R is noetherian, then Rx, the homomorphic image of R, is noetherian.

Now consider the family

$\mathfrak{I}$ = family of all submodules of Rx which intersect at least one submodule of Rx trivially.

As $(0) \in \mathfrak{I}$, $\mathfrak{I} \neq \phi$. Now Rx is Noetherian implies that $\mathfrak{I}$ has a maximal element say, K. By definition of $\mathfrak{I}$, there exists a submodule U of Rx such that $K \cap U = (0)$.

Now we claim that U is uniform.

Suppose if possible, there exist submodules A and B of U such that $A \cap B = (0)$.

Now $K \cap A \subseteq K \cap U = (0)$.

Consider $(K \oplus A) \cap B$.

Let x belong to this intersection.

Then $x = y + a = b$, where $y \in K$, $a \in A$ and $b \in B$.

This implies $y = b - a \in K \cap U = (0)$

$\Rightarrow \quad y = 0$ and $b - a = 0$

$\Rightarrow \quad x = a = b \in A \cap B = (0)$

$\Rightarrow \quad x = 0.$

Hence, $(K \oplus A) \cap B = (0)$. This implies $K \oplus A \in \mathfrak{I}$, a contradiction to the maximality of K.

Hence, U is uniform submodule of Rx.

Hence proved.

Definition 4. Let R be a commutative Noetherian ring and P be a prime ideal of R. Then **P is said to be associated with an R-module M** if R/P embeds in M. Equivalently, P = $Ann_R(x)$ for some $x \in M$, where $Ann_R(x) = \{r \in R \mid rx = 0\}$ is the annihilator of x in R.

Definition 5. A module M is called **P-primary** for some prime ideal P if P is the only prime ideal associated with M.

Theorem 2. Let U be a uniform module over a commutative Noetherian ring R. Then U contains a submodule isomorphic to R/P for precisely one prime ideal P, that is, U is subisomorphic to R/P for exactly one prime ideal P of R.

Proof. Let $\mathfrak{I} = \{Ann_R(x) \mid 0 \neq x \in U\}$ be the family of annihilators of non-zero $x \in U$ in R. As R is Noetherian $\mathfrak{I}$ has a maximal element, say, $Ann_R(x)$ for $0 \neq x \in U$.

Now, we claim that $Ann_R(x) = P$(say) is prime.

Clearly, P is an ideal of R.

Now let $ab \in P$. Then $abx = 0$.

Suppose $b \notin P$, that is, $bx \neq 0$.

Now $P = Ann_R(x) \subseteq Ann_R(bx)$ as R is commutative.

By maximality of P, we get $P = Ann_R(bx)$.

As $a \in Ann_R(bx)$, $a \in P$.

Thus P is prime.

Also $R/P \cong Rx \subseteq U$ and hence U is subisomorphic to R/P.

Now we prove the uniqueness.

Suppose Q is a prime ideal of R such that U is subisomorphic to R/Q.

Then [R/Q] = [U] = [R/P]

$\Rightarrow$ R/Q is subisomorphic to R/P.

Thus, there exist cyclic submodules Rx and Ry of R/P and R/Q such that $Rx \cong Ry$.

As $Rx \cong R/P$ and $Ry \cong R/Q$, we get

$$R/P \cong R/Q \Rightarrow P = Q.$$

Hence proved.

Theorem 3. Let M be a non-zero finitely generated module over a commutative Noetherian ring R. Then there are only finitely many prime associated with M.

Proof. Define $\mathcal{F}$ to be the family of direct sums of cyclic uniform submodules of M.

By Theorem 1, $\mathcal{F} \neq \phi$.

Define partial order in $\mathcal{F}$ by

$$\oplus\sum_{i\in I} x_i R \leq \oplus\sum_{j\in J} y_j R \text{ if and only if } I \subseteq J \text{ and } x_iR \subseteq y_iR \text{ for all } i \in I.$$

By Zorn's Lemma, $\mathcal{F}$ has a maximal element, say, $N = \oplus\sum_{\lambda\in J} x_\lambda R$.

As M is Noetherian, N is finitely generated and hence $N = \oplus\sum_{i=1}^{m} x_i R$ for some positive integer m.

As each x_iR is uniform, by Theorem 2, there exist $x_ia_i \in x_iR$ such that $P_i = Ann_R(x_ia_i)$ is the prime ideal associated with x_iR.

Let $K = \sum_{i=1}^{m} x_i a_i R$.

Now, we claim that if Q is any associated prime ideal of M, then Q = P_i for some i, $1 \le i \le m$.

Now Q = $\text{Ann}_R(x)$ for some $x \in M$.

As N is a maximal member of $\mathcal{F}$, both N and K intersect every non-zero submodule of M non-trivially.

Let $0 \neq y \in xR \cap K$. Then $y = xr = \sum_{i=1}^{m} x_i a_i r_i$.

Let $x_i a_i r_i s = 0 \Rightarrow r_i s \in \text{Ann}_R(x_i a_i) = P_i$.

Suppose $x_i a_i r_i \neq 0$.

Then $r_i \notin P_i$.

This gives $s \in P_i$.

Hence if $x_i a_i r_i \neq 0$, $\text{Ann}_R(x_i a_i) = \text{Ann}_R(x_i a_i r_i)$.

Now $\text{Ann}_R(y) = \bigcap_{i=1}^{m} \text{Ann}_R(x_i a_i r_i)$

$$= \bigcap_{i \in \Lambda} \text{Ann}_R(x_i a_i)$$

$$= \bigcap_{i \in \Lambda} P_i, \text{ where } i \in \Lambda \text{ implies } x_i a_i r_i \neq 0.$$

Now $R/Q \cong_\theta xR$. As $\theta^{-1}(yR)$ is cyclic submodule of R/Q, $\theta^{-1}(yR) \cong R/Q$.

Now $Q = \text{Ann}_R(x) = \text{Ann}_R(y) = \bigcap_{i \in \Lambda} P_i$.

Thus $Q \subseteq P_i \ \forall \ i \in \Lambda$.

Now suppose $P_i \not\subseteq Q \ \forall \ i \in \Lambda$. Then there exist $x_i \in P_i$ such that

$x_i \notin Q \ \forall \ i \in \Lambda$.

As $\prod_{i\in\Lambda} x_i \in \bigcap_{i\in\Lambda} P_i = Q$ and Q is prime, this leads to a contradiction.

Hence $P_i \subseteq Q$ for some $i \in \Lambda$.

Thus, $Q = P_i$ for some i = 1, 2, …, m.

Hence proved.

Noether-Lasker Theorem :

Let M be a finitely generated module over a commutative Noetherian ring R. Then there exists a finite family $N_1, N_2, \ldots, N_m$ of submodules of M such that

(a) $\bigcap_{i=1}^{m} N_i = 0$ and $\bigcap_{i\neq j} N_i \neq 0 \quad \forall\ 1 \leq j \leq m$.

(b) Each quotient M/N_i is P_i–primary module for some prime ideal P_i.

(c) The P_i's are all distinct.

(d) The primary components N_i is unique if and only if P_i does not contain P_j for $j \neq i$.