Example 3. For any group G, the identity mapping $i : G \to G$ is an automorphism.

Example 4: Let G be the group $(R^+, \cdot)$ of positive real numbers under multiplication and let H be the additive group R. Then the mapping $\phi : R^+ \to R$ given by $\phi(x) = \log x$ is an isomorphism.

Example 5: Let G be a group. For a given 'a' $\in G$ consider the mapping $I_a : G \to G$ given by

$$I_a(x) = axa^{-1} \quad \forall\, x \in G.$$

Then I_a is an automorphism of G called the inner automorphism of G determined by 'a'.

Proof. Given a mapping $I_a : G \to G$ defined by $I_a(x) = axa^{-1}$ — (1)

To show that $I_a : G \to G$ is an automorphism, we will show that

(i) $I_a : G \to G$ is a homomorphism,

(ii) $I_a : G \to G$ is one-one and

(iii) $I_a : G \to G$ is onto.

(i) To show that $I_a : G \to G$ is homomorphism, we have to prove that

$$I_a(xy) = I_a(x) \cdot I_a(y).$$

For $x, y \in G$, consider

$$\begin{aligned}
I_a(xy) &= axya^{-1} && (\text{by } (1)) \\
&= axeya^{-1} && (\because e \text{ is identity in } G) \\
&= ax(a^{-1}a)ya^{-1} && (\because e = a^{-1}a)
\end{aligned}$$

$$= (axa^{-1}).(aya^{-1})$$

$$= I_a(x).I_a(y) \quad (\text{by } ①)$$

i.e. $I_a(xy) = I_a(x).I_a(y)$

$\Rightarrow I_a : G \to G$ is a homomorphism.

(ii) To prove that I_a is one-one, we have to show that whenever $I_a(x) = I_a(y) \Rightarrow x = y$.

Consider
$$I_a(x) = I_a(y)$$
$$\Rightarrow \quad axa^{-1} = aya^{-1} \quad (\text{by } ①)$$
$$\Rightarrow \quad x = y \quad (\text{by cancellation laws})$$

Hence, $I_a : G \to G$ is injective.

(iii) For any x in G, consider
$$x = aa^{-1}xaa^{-1}$$
$$= a(a^{-1}xa)a^{-1}$$
$$= I_a(a^{-1}xa) \quad (\text{by } ①)$$
$$= I_a(x) \in G$$

$\Rightarrow I_a$ is onto.

Hence, $I_a : G \to G$ is an automorphism.

Note: The set of all inner automorphisms of G is denoted by $I_n(G)$.

The set of all automorphisms of G is denoted by $\text{Aut}\, G$, or

$$\text{Aut}(G) = \{\pi \mid \pi : G \to G \text{ is an automorphism}\}$$

Theorem: If G is a group, then $Aut(G)$ is a group with composition of maps as binary operation.

Proof. To show that $Aut(G)$ is a group with composition of maps, we show that

(1) $Aut(G)$ is closed under composition.

(2) Associative law.

(3) Existence of identity.

(4) Existence of inverse.

(1) Closure:

Suppose $\sigma, \varsigma \in Aut(G)$ then by definition, $\sigma : G \to G$ and $\varsigma : G \to G$ is an automorphism, i.e. ς and σ are isomorphism. Then, we have to show that

$$\sigma\varsigma \in Aut(G).$$

We know that if two mappings are bijective then their composite mapping is also bijective.

$\therefore$ $\sigma\varsigma$ is also bijective.

Now, to show that $\sigma\varsigma$ is homomorphism, consider for $x, y \in G$,

$$\begin{aligned} \sigma\varsigma(xy) &= \sigma(\varsigma(xy)) \\ &= \sigma(\varsigma(x).\varsigma(y)) \quad (\because \varsigma \text{ is an automorphism}) \\ &= \sigma(\varsigma(x)).\sigma(\varsigma(y)) \quad (\because \sigma \text{ is an automorphism}) \end{aligned}$$

i.e. $\sigma\varsigma(xy) = \sigma\varsigma(x).\sigma\varsigma(y)$

$\Rightarrow$ $\sigma\varsigma$ is homomorphism.

Hence, $\sigma\varsigma \in Aut(G)$.

$\Rightarrow$ $Aut(G)$ is closed under composition.

(2) Associative law:

Consider $\sigma, \varsigma, \eta \in Aut(G)$.

Then we have to show that
$$(\sigma \circ \rho) \circ \eta = \sigma \circ (\rho \circ \eta).$$
Now, for any $x \in G$, consider
$$(\sigma \rho)\eta(x) = \sigma\rho(\eta(x))$$
$$= \sigma(\rho(\eta(x)))$$
$$= \sigma(\rho\eta(x))$$
$$= \sigma(\rho\eta)(x)$$
i.e. $(\sigma \circ \rho) \circ \eta = \sigma \circ (\rho \circ \eta)$

$\Rightarrow$ Associative law holds in $Aut(G)$.

(3) Existence of identity: Suppose $\sigma \in Aut(G)$.

Consider the map $I : G \to G$ given by
$$I(x) = x.$$
Then we have to prove that
$$\sigma \circ I = \sigma$$
or $(\sigma \circ I)(x) = \sigma(x)$

Consider,
$$(\sigma \circ I)(x) = \sigma I(x)$$
$$= \sigma(x) \qquad (\because I \text{ is identity map})$$
$\Rightarrow$ Identity holds in $Aut(G)$.

(4) Existence of inverse:

Suppose $\sigma \in Aut(G)$, then σ is bijective and hence σ is invertible. ($\because$ We know that "a mapping f is invertible iff f is bijective")

i.e. σ^{-1} exists.

Now, we have to show that $\sigma^{-1} \in Aut(G)$.

Now, $\sigma^{-1} : G \to G$ is defined as
$$\sigma^{-1}(x) = y \quad \text{iff} \quad \sigma(y) = x.$$

Now, if $x, y \in G$ then $g, h \in G$ such that

$$\sigma(g) = x \text{ and } \sigma(h) = y \text{ and so}$$

$$\begin{aligned}\sigma^{-1}(xy) &= \sigma^{-1}(\sigma(g).\sigma(h)) \\ &= \sigma^{-1}(\sigma(gh)) \quad (\because \sigma \text{ is homomorphism}) \\ &= gh \\ &= \sigma^{-1}(x).\sigma^{-1}(y).\end{aligned}$$

$\Rightarrow \sigma^{-1}: G \to G$ is a homomorphism.

Obviously, $\sigma^{-1}: G \to G$ is 1-1 and onto.

Hence, $\sigma^{-1} \in Aut(G)$.

This proves the result.

Definition. (Centre of a group G):

The centre of a group G, written as $Z(G)$, is the set of those elements in G that commute with every element in G, i.e.

$$Z(G) = \{a \in G \mid ax = xa \ \forall x \in G\}.$$

Definition. (Kernel):

Suppose G and H be two groups and let $\phi: G \to H$ be a homomorphism. Then Kernel of ϕ is defined to be the set

$$Ker\,\phi = \{x \in G \mid \phi(x) = e'\},$$

where e' is identity in H.

Note: Suppose $\phi: G \to H$ be a homomorphism of groups. Then $Ker\,\phi$ is a subgroup of G and $Im\,\phi$ is a subgroup of H, where $Im\,\phi$ is image of ϕ.