M.Sc. II Sem. (Mathematics)

Paper 1st - Advanced Abstract Algebra-II

Unit - III

Reference Book :
- **P.B.Bhattacharya, S.K. Jain and S.R Nagpaul, *Basic Abstract Algebra* (2nd Edition), Cambridge University Press, Indian Edition, 1997.**
- **C. Musili, *Introduction to Rings and Modules,* Second Revised Edition, Narosa Publishing House, New Delhi.**

Topic : Noetherian and Artinian Modules

Thoerem : Every homomorphic image of a noetherian module is also a noetherian module.

Proof. Suppose M_1 be a noetherian module.

Suppose $f : M_1 \rightarrow M_2$ be a surjective homomorphism, where M_1 and M_2 are modules.

Then we have to show that M_2 is a noetherian module.

Suppose B be the submodule of M_2.

Then $f^{-1}(B)$ is the submodule of M_1.

Since M_1 is noetherian and so $f^{-1}(B)$ is finitely generated.

Since the image of a finitely generated submodule is finitely generated and so

$f\left(f^{-1}(B)\right) = B$, which is finitely generated. $(\because$ B is surjective)

Since B is arbitrary, and so every submodule of M_2 is finitely generated and so M_2 is noetherian. ($\because$ If every submodule of M is finitely generated, then M is noetherian)

Hence, every homomorphic image of a noetherian module is also a noetherian module.

Hence proved.

Corollary : Suppose M be a noetherian module and L be a submodule of M. Then M/L is also noetherian.

Proof. Since quotient module M/L is homomorphic image of M, hence by theorem "Every homomorphic image of a noetherian module is also a noetherian module", we conclude that M/L is also noetherian.

Hence proved.

Theorem : Suppose M be an R-module and N be a submodule of M such that N and M/N are noetherian then M is also noetherian.

Proof. To show that M is noetherian, consider an ascending chain of submodules of M :

$$M_1 \subset M_2 \subset M_3 \subset \ldots \subset M_n \subset M_{n+1} \subset \ldots .$$

Intersecting with N gives an ascending chain

$$N \cap M_1 \subset N \cap M_2 \subset N \cap M_3 \subset \ldots \subset N \cap M_n \subset N \cap M_{n+1} \subset \ldots .$$

Since N is noetherian module, then there exists a number r such that

$$N \cap M_r = N \cap M_{r+1}.$$

Also, M/N is noetherian module and so there exists an ascending chain

$$\frac{N+M_1}{N} \subset \frac{N+M_2}{N} \subset \dots \subset \frac{N+M_r}{N} \subset \dots$$

such that for some s

$$\frac{N+M_s}{N} = \frac{N+M_{s+1}}{N}.$$

From above result, we have

(i) $M_n \subset M_{n+1}$

(ii) $N \cap M_n \subset N \cap M_{n+1}$ for $n \geq r$

(iii) $\frac{N+M_n}{N} = \frac{N+M_{n+1}}{N}$ for some $n \geq s$.

From second isomorphism theorem for modules, we have

$$\frac{N+M_n}{N} \simeq \frac{M_n}{N \cap M_n} \quad \forall\, n \in N.$$

Hence, we get

$$\frac{M_n}{N \cap M_n} = \frac{N+M_n}{N} = \frac{N+M_{n+1}}{N} = \frac{M_{n+1}}{N \cap M_{n+1}} \quad \text{for } n \geq r+s.$$

Hence for $n \geq r + s$, we get

$M_n = M_{n+1}$.

Hence, M satisfy ascending chain condition and so M is noetherian.

Hence proved.

Some Pathologies:

1. An Artinian module need not be finitely generated.
2. Maximal submodules need not exist in an Artinian module.
3. An Artinian module need not be Noetherian.
4. A finitely generated module need not be Noetherian.
5. Minimal submodules need not exist in a Noetherian module.
6. A Noetherian module need not be Artinian.
7. There are modules which are neither Artinian nor Noetherian.

Definition : A ring R is called a **left noetherian ring** if R regarded as a left R-module is noetherian. Similarly, a ring R is called a **right noetherian ring** if R regarded as a right R-module is noetherian.

Definition : A ring R is called a **left artinian ring** if R regarded as a left R-module is artinian. Similarly, a ring R is called a **right artinian ring** if R regarded as a right R-module is artinian.

Examples :

(a) Let V be an n-dimensional vector space over a field F. Then V is both noetherian and artinian.

(b) Let A be a finite-dimensional algebra with unity over a field F. Then A as a ring is both left and right noetherian as well as artinian.

(c) Let R = F[x] be a polynomial ring over a field F in x. Then F[x] is a noetherian ring.

(d) Finite abelian groups are Noetherian as modules over Z.

(e) Finite dimensional vector spaces are Noetherian (for dimension reasons) whereas infinite dimensional ones are not Noetherian.

(f) Unlike the Artinian case, infinite cyclic groups are Noetherian.

Theorem. A subring of a noetherian (artinian) ring need not be noetherian (artinian).

Proof. For the artinian case the ring of rational numbers Q is an artinian ring, but its subring Z is not an artinian ring.

For the noetherian case, the ring of 2×2 matrices over the rational numbers Q is a noetherian ring, but its subring $\begin{bmatrix} Z & Q \\ 0 & Q \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \; a \in Z, \; b, c \in Q \right\}$ is not noetherian, that is, not left noetherian.

Hilbert Basis Theorem :

Let R be a noetherian ring. Then the polynomial ring R[x] is also a noetherian ring.

Proof. Suppose there exists an ideal $I \subset R[x]$, which is not finitely generated.

Set $I_0 = (0)$, then let $f_1 \in I$ be a polynomial in I of least degree.

Let I_1 be the ideal generated by f_1, i.e.

$$I_1 = (f_1).$$

Let f_2 be an element of least degree in $I \setminus (f_1)$.

Let f_2 be an element of least degree in $I \setminus (f_1, f_2)$

and $I_2 = (f_1, f_2)$.

Recursively, let f_i be a polynomial of least degree in $I \setminus (f_1, \ldots, f_{i-1})$.

Then we observe that

(1) $\deg f_1 \le \deg f_2 \le \deg f_3 \le \ldots$

(2) $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$.

Let a_i be the leading coefficient of f_i.

Let $J_i = (a_1, \ldots, a_i)$

$\Rightarrow J_1 \subseteq J_2 \subseteq J_3 \subseteq \ldots$

Since this is a chain of ideals in R, which is noetherian, and so there exists n such that

$$J_n = J_{n+1} = \ldots$$

$\Rightarrow a_{n+1} = \sum_{i=1}^{n} r_i a_i$ for some $r_i \in A$.

Let $f = f_{n+1} - \sum_{i=1}^{n} r_i f_i$.

$\Rightarrow$ $\deg f < \deg f_{n+1}$ and $f \in I_{n+1}$

$\Rightarrow$ $f \in I_n$. [Since f_{n+1} is a polynomial of least degree in $I \setminus (f_1, \ldots, f_n)$]

Therefore, $f_{n+1} \in I_n$, which is a contradiction.

$\Rightarrow$ I is finitely generated.

Thus, R[x] is noetherian.

Hence proved.