Example 2: Let $f(x;\theta) = \frac{1}{\pi} \frac{1}{[1+(x-\theta)^2]}$, $-\infty < x < \infty$

Taking logarithm and then differentiating with respect to θ we have

$$\frac{\partial}{\partial \theta} \log f(x_1, x_2, \dots, x_n; \theta) = 2 \sum_{i=1}^{n} \frac{(x_i - \theta)}{1+(x_i-\theta)^2}$$

Since it is not of the form $A(\theta).[T-\tau(\theta)]$, we conclude that, whatever it is, the minimum variance unbiased estimate of the parameter of this distribution does not attain the Cramer-Rao lower bound.

Example 3: Consider estimating the parameter λ of Poisson distribution having pmf

$$f(x;\lambda) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

Here,

$$\begin{aligned} \frac{\partial \log f(x_1, \dots, x_n; \lambda)}{\partial \lambda} &= -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} \\ &= \frac{n}{\lambda} (\bar{x} - \lambda) \\ &\simeq A(\lambda)(\hat{\lambda} - \lambda), \end{aligned}$$

where $A(\lambda) = \frac{n}{\lambda}$ and $\hat{\lambda} = \bar{x}$. Since $\bar{X}$ is unbiased, it is an estimator whose variance attains the Cramer-Rao lower bound i.e. $\bar{X}$ is MVUE of λ.

Condition for equality in the CR inequality

Theorem: The variance of an unbiased estimator T of $\tau(\theta)$ actually attains the Cramer-Rao bound if and only if $\frac{\partial}{\partial\theta}\log f(x_1,\ldots,x_n;\theta)$ can be written in the form

$$\frac{\partial}{\partial\theta}\log f(x_1,\ldots,x_n;\theta) = A(\theta)\{T-\tau(\theta)\}$$

where $A(\theta)$ is independent of $x_1, x_2, \ldots, x_n$ but may depend on θ.

Proof: The inequality

$$V(T) \geq \frac{[\tau'(\theta)]^2}{E\left[\frac{\partial}{\partial\theta}\log f(x_1,\ldots,x_n;\theta)\right]^2}$$

arises by the use of Schwarz inequality

$$[\mathrm{Cov.}(U,V)]^2 \leq V(U).V(V).$$

The necessary and sufficient condition that the Schwarz inequality be an equality is that one variable is a linear function of another r.v. with probability one. Here for equality $T-\tau(\theta)$ should be proportional to $\frac{\partial}{\partial\theta}\log f(x_1,\ldots,x_n;\theta)$ for all possible $x_1,\ldots,x_n$. Thus CR inequality can be equality if $\frac{\partial}{\partial\theta}\log f(x_1,\ldots,x_n;\theta)$ can be written as

$$\frac{\partial}{\partial\theta}\log f(x_1,\ldots,x_n;\theta) = A(\theta)\,[T-\tau(\theta)] \quad \ldots(1)$$

for $A(\theta)$ independent of $x_1,\ldots,x_n$ but may depend on θ.

Multiplying both sides by $T-\tau(\theta)$ and taking expectation we get (from eqn (5) of CR inequality)

$$\tau'(\theta) = E\left[\frac{\partial}{\partial\theta}\log f(x_1,\ldots,x_n;\theta)\{T-\tau(\theta)\}\right] = A(\theta).V(T)$$

$$\Rightarrow V(T) = \frac{\tau'(\theta)}{A(\theta)} \quad \ldots(2)$$

Further if $\tau(\theta)=\theta$, then $V(T) = 1/A(\theta)$

Thus in situations where the Cramer-Rao lower bound can be reached, the minimum variance unbiased estimator can be found directly from this theorem. If (1) is satisfied, then T is a MVB estr of $\tau(\theta)$ ... $-[\tau'(\theta)]^2/E\left[\frac{\partial^2}{\partial\theta^2}\log f(x_1,\ldots,x_n;\theta)\right]$

CBCS Course

S.S. in Statistics

Vikram University, Ujjain

Class : MA/MSc Second Semester

Subject : Statistics

Paper-I : Statistical Inference - I

Unit-II : Lecture 4

Title : Condition for equality in CR Inequality

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