Theorem (Cramer-Rao Inequality)

In any regular estimation case, the variance of an unbiased estimator T for $\tau(\theta)$, which is assumed to be a differentiable function of θ, satisfies the inequality

$$V(T) \geq \frac{[\tau'(\theta)]^2}{E[\frac{\partial}{\partial\theta} \log f(x_1,\dots,x_n;\theta)]^2} \quad \dots(1)$$

Proof We have

$$1 = \int_A \dots \int f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n$$

Differentiating it with respect to θ and using assumption (ii), we have

$$0 = \int_A \dots \int \frac{\partial}{\partial\theta} f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n$$

or $$0 = \int_A \dots \int \frac{\partial}{\partial\theta}[\log f(x_1,\dots,x_n;\theta)] . f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n \quad \dots(2)$$

$$\Rightarrow E[\frac{\partial}{\partial\theta} \log f(x_1,\dots,x_n;\theta)] = 0$$

Again T being an unbiased estimator of $\tau(\theta)$,

$$\tau(\theta) = \int_A \dots \int t . f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n = E[T]$$

Differentiating it with respect to θ and using assumption iv) we have

$$\tau'(\theta) = \frac{\partial}{\partial\theta}\tau(\theta) = \int_A \dots \int t . \frac{\partial}{\partial\theta} f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n$$

$$= \int_A \dots \int t \frac{\partial \log f(x_1,\dots,x_n;\theta)}{\partial\theta} . f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n \quad \dots(3)$$

Multiplying (2) by $\tau(\theta)$ and subtracting the product from (3), we get

$$\tau'(\theta) = \int_A \dots \int \{t-\tau(\theta)\} \{\frac{\partial}{\partial\theta} \log f(x_1,\dots,x_n;\theta)\} f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n.$$

$$= E[\{T-\tau(\theta)\}\{\frac{\partial}{\partial\theta} \log f(x_1,\dots,x_n;\theta) - 0\}]$$

$$= Cov.[T, \frac{\partial}{\partial\theta} \log f(x_1,\dots,x_n;\theta)]$$ since from (2) $E[\frac{\partial}{\partial\theta} \log f(x_1,\dots,x_n;\theta)] = 0$

Here T has expectation $\tau(\theta)$ and from (2) the expectation of $\frac{\partial}{\partial\theta} \log f(x_1,\dots,x_n;\theta)$ is zero, Hence by Cauchy-Schwarz inequality $[Cov.(X,Y)]^2 \leq V(X).V(Y)$, we have by virtue of assumption v)

$$[\tau'(\theta)]^2 \leq V(T). E[\frac{\partial}{\partial\theta} \log f(x_1,\dots,x_n;\theta)]^2$$ giving the desired result.

Regularity Conditions for Cramer-Rao Inequality

The Cramer-Rao inequality was presented independently and almost simultaneously by Rao, Cramer, Darmois and Frechet.

Suppose θ is a single parameter varying over the parameter space Θ and that $X_1, X_2, \ldots, X_n$ are all continuous variables with joint pdf $f(x_1, x_2, \ldots, x_n; \theta)$. We make the following assumptions:

i) Θ is a non-degenerate open interval on the real line.

ii) For almost all $x_1, x_2, \ldots, x_n$ and all $\theta \in \Theta$

$$\frac{\partial}{\partial\theta} f(x_1, x_2, \ldots, x_n; \theta)$$

exists, the exceptional, if any, being independent of θ

iii) $\frac{\partial}{\partial\theta} \int_A \cdots \int f(x_1, \ldots, x_n; \theta)\, dx_1 \ldots dx_n = \int_A \cdots \int \frac{\partial}{\partial\theta} f(x_1, \ldots, x_n; \theta)\, dx_1 \ldots dx_n$

'A' denoting the domain of positive probability density.

iv) $\frac{\partial}{\partial\theta} \int_A \cdots \int t(x_1, \ldots, x_n)\, f(x_1, \ldots, x_n; \theta)\, dx_1 \ldots dx_n$

$$= \int_A \cdots \int t(x_1, \ldots, x_n) \frac{\partial}{\partial\theta} f(x_1, \ldots, x_n; \theta)\, dx_1 \ldots dx_n$$

v) $E\left[\frac{\partial}{\partial\theta} \log f(x_1, \ldots, x_n; \theta)\right]^2$ exists and is positive for every $\theta \in \Theta$.

Here, assumption (ii) and (v) require that the domain of positive

Theorem: (Cramer-Rao Inequality)

In any regular estimation case, the variance of an unbiased estimator T for $\tau(\theta)$, which is assumed to be a differentiable function of θ, satisfies the inequality

$$V(T) \geqslant \frac{[\tau'(\theta)]^2}{E\left[\frac{\partial}{\partial\theta}\log f(x_1,\dots,x_n;\theta)\right]^2} \qquad \dots(1)$$

Proof: We have

$$1 = \int_A \dots \int f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n$$

Differentiating it with respect to θ and using assumption (ii), we have

$$0 = \int_A \dots \int \frac{\partial}{\partial\theta} f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n$$

or

$$0 = \int_A \dots \int \frac{\partial}{\partial\theta}[\log f(x_1,\dots,x_n;\theta)] . f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n \qquad \dots(2)$$

$$\Rightarrow E\left[\frac{\partial}{\partial\theta}\log f(x_1,\dots,x_n;\theta)\right] = 0$$

Again T being an unbiased estimator of $\tau(\theta)$,

$$\tau(\theta) = \int_A \dots \int t . f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n = E[T]$$

Differentiating it with respect to θ and using assumption iv) we have

$$\tau'(\theta) = \frac{\partial}{\partial\theta}\tau(\theta) = \int_A \dots \int t . \frac{\partial}{\partial\theta} f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n$$

$$= \int_A \dots \int t\, \frac{\partial}{\partial\theta}\log f(x_1,\dots,x_n;\theta) . f(x_1,\dots,x_n;\theta)\, dx_1 \dots dx_n \qquad \dots(3)$$

Multiplying (2) by $\tau(\theta)$ and subtracting the product from (3), we get

$$\tau'(\theta) = \int_A \dots \int \{t-\tau(\theta)\}\left\{\frac{\partial}{\partial\theta}\log f(x_1,\dots,x_n;\theta)\right\} f(x_1,\dots,x_n,\theta)\, dx_1 \dots dx_n.$$

$$= E\left[\{T-\tau(\theta)\}\left\{\frac{\partial}{\partial\theta}\log f(x_1,\dots,x_n;\theta)-0\right\}\right]$$

$$= \mathrm{Cov.}\left[T, \frac{\partial}{\partial\theta}\log f(x_1,\dots,x_n;\theta)\right] \quad \text{since from (2) } E\left[\frac{\partial}{\partial\theta}\log f(x_1,\dots,x_n;\theta)\right] = 0$$

Here T has expectation $\tau(\theta)$ and from (2) the expectation of $\frac{\partial}{\partial\theta}\log f(x_1,\dots,x_n;\theta)$ is zero. Hence by Cauchy-Schwarz inequality $[\mathrm{Cov.}(X,Y)]^2 \leq V(X).V(Y)$, we have by virtue of assumption v)

$$[\tau'(\theta)]^2 \leq V(T) . E\left[\frac{\partial}{\partial\theta}\log f(x_1,\dots,x_n;\theta)\right]^2$$ giving the desired result.

C B C S. Course

M. Sc. Statistics

Second Semester

Paper - I. Statistical Inference

Unit - II Title.

"CRAMER - RAO INEQUALITY".

Teacher : Dr. Praveen Gupta