Class Name - M. Sc. (STATISTICS)

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Title of Lecture - Bivariate Discrete Random Variables and Distributions

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• BIVARIATE DISCRETE RANDOM VARIABLES

• **Definition** – Let X and Y be two discrete random variables defined on the sample space S of a random experiment then the function (X, Y) defined on the same sample space is called a two-dimensional discrete random variable. In other words, (X, Y) is a two-dimensional discrete random variable if the possible values of (X, Y) are finite or countably infinite. Here, each value of X and Y is represented as a point (x, y) in the xy-plane.

• **Definition** – Let X and Y be two functions defined on a discrete probability space. Let R denote the corresponding two-dimensional space of X and Y, the two random variables of the discrete type. The probability that $X = x_i$ and $Y = y_j$ is denoted by $p_{ij} = P[X = x_i, Y = y_j] = P[X = x_i \cap Y = y_j] = p(x_i, y_j)$ and it is induced from the discrete probability space through the functions X and Y. The function $p(x_i, y_j)$ is called the joint probability mass function of X and Y and has the following properties:

(i) $0 \leq p(x_i, y_j) \leq 1$

(ii) $\sum_i \sum_j p(x_i, y_j) = 1$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, 3, \ldots, m$.

It is usually represented in the form of the Table

X \ Y	y_1	y_2	y_3	$\cdots$	y_j	$\cdots$	y_m	Total
x_1	p_{11}	p_{12}	p_{13}	$\cdots$	p_{1j}	$\cdots$	p_{1m}	$p_{1\cdot}$
x_2	p_{21}	p_{22}	p_{23}	$\cdots$	p_{2j}	$\cdots$	p_{2m}	$p_{2\cdot}$
x_3	p_{31}	p_{32}	p_{33}	$\cdots$	p_{3j}	$\cdots$	p_{3m}	$p_{3\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_i	p_{i1}	p_{i2}	p_{i3}	$\cdots$	p_{ij}	$\cdots$	p_{im}	$p_{i\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	p_{n1}	p_{n2}	p_{n3}		p_{nj}	$\cdots$	p_{nm}	$p_{n\cdot}$
Total	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	$\cdots$	$p_{\cdot j}$	$\cdots$	$p_{\cdot m}$	1

$$\therefore \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j) = 1$$

Suppose the joint distributions of two random variables X and Y is given, then the probability distribution of X is determined as follows:

$$p(x_i) = P[X=x_i] = P\left[(X=x_i \cap Y=y_1) \text{ or } (X=x_i \cap Y=y_2) \text{ or } (X=x_i \cap Y=y_3) \text{ or } \cdots\right]$$

$$= P[X=x_i \cap Y=y_1] + P[X=x_i \cap Y=y_2] + P[X=x_i \cap Y=y_3] + \cdots + P[X=x_i \cap Y=y_m]$$

$$= \sum_{j}^{m} P[X=x_i \cap Y=y_j]$$

$$= \sum_{j}^{m} p(x_i, y_j) \quad \left[\because p(x_i, y_j), \text{ the joint probability mass function is } P[X=x_i \cap Y=y_j]\right] \quad (1)$$

$$= \sum_{j=1}^{m} p_{ij}$$

$$= p_{i\cdot}$$

which is termed as the marginal probability mass function of X

Also $\sum_{i=1}^{n} p_{i\cdot} = p_{1\cdot} + p_{2\cdot} + \cdots + p_{n\cdot} = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij} = 1$

Similarly, we can prove that

$$p(y_j) = P[Y=y_j] = P[X=x_1 \cap Y=y_j] + P[X=x_2 \cap Y=y_j] + \cdots + P[X=x_n \cap Y=y_j]$$
$$= \sum_{i=1}^{n} P[X=x_i \cap Y=y_j] = \sum_{i=1}^{n} p(x_i, y_j)$$
$$= \sum_{i=1}^{n} p_{ij}$$
$$= p_{\cdot j} \qquad (2)$$

which is the marginal probability mass function of Y.

• CONDITIONAL PROBABILITY MASS FUNTION-

Let (X, Y) be a discrete two-dimensional random variable. Then the conditional probability mass function of X, given Y=y is given by

$$p(x_i|y_j) = P[X=x_i | Y=y_j] = \frac{P[X=x_i \cap Y=y_j]}{P[Y=y_j]} = \frac{p(x_i, y_j)}{p(y_j)}$$
$$= \frac{p_{ij}}{p_{\cdot j}} \qquad (3)$$

provided that $P[Y=y_j] \neq 0$.

Similarly

$$p(y_j|x_i) = P[Y=y_j | X=x_i] = \frac{P[X=x_i \cap Y=y_j]}{P[X=x_i]} = \frac{p(x_i,}{p(x_i)} = \frac{p_{ij}}{p_{i\cdot}} \qquad (4)$$

is the conditional probability mass function of Y given $X=x_i$.

Further, we have

$$\sum_{i=1}^{n} \frac{p_{ij}}{p_{\cdot j}} = \frac{p_{1j} + p_{2j} + \cdots + p_{ij} + \cdots + p_{nj}}{p_{\cdot j}} = \frac{p_{\cdot j}}{p_{\cdot j}} = 1 \qquad (5)$$

Similarly

$$\sum_{j=1}^{n} \frac{p_{ij}}{p_{i\cdot}} = \frac{p_{i1} + p_{i2} + \cdots + p_{ij} + \cdots + p_{im}}{p_{i\cdot}} = \frac{p_{i\cdot}}{p_{i\cdot}} = 1 \qquad (6)$$

• INDEPENDENCE OF RANDOM VARIABLES –

Two discrete random variables X and Y are said to be independent if and only if

$$P[X=x_i \cap Y=y_j] = P[X=x_i]\, P[Y=y_j]$$

[∵ Two events A and B are independent if and only if $P(A\cap B)=P(A)P(B)$]

• Example-1 – The following table represents the joint probability distribution of the discrete random variable (X, Y):

X \ Y	1	2
1	0.1	0.2
2	0.1	0.3
3	0.2	0.1

Find (i) The marginal distributions.
(ii) The conditional distribution of X given Y=1.
(iii) $P[(X+Y)<4]$

Solution – (i) To find the marginal distributions, we have to find the marginal totals, i.e. row totals and column totals as shown in the following table:

X \ Y	1	2	p(x) (Totals)
1	0.1	0.2	0.3
2	0.1	0.3	0.4
3	0.2	0.1	0.3
p(y) (Totals)	0.4	0.6	1.0

Thus the marginal probability distribution of X is

X	1	2	3
p(x)	0.3	0.4	0.3

and the marginal probability distribution of Y is

Y	1	2
p(y)	0.4	0.6

(ii) As $P[X=1|Y=1] = \dfrac{P[X=1, Y=1]}{P[Y=1]} = \dfrac{0.1}{0.4} = \dfrac{1}{4}$

$P[X=2|Y=1] = \dfrac{P[X=2, Y=1]}{P[Y=1]} = \dfrac{0.1}{0.4} = \dfrac{1}{4}$ and

$P[X=3|Y=1] = \dfrac{P[X=3, Y=1]}{P[Y=1]} = \dfrac{0.2}{0.4} = \dfrac{2}{4} = \dfrac{1}{2}$.

Therefore the conditional distribution of X given Y=1 is

x	1	2	3
$P[X=x\|Y=1]$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

(iii) It is observed from the joint distribution table that the values of (X, Y) which satisfy $X+Y<4$ are (1,1), (1,2) and (2,1) only.

Therefore

$$P[(X+Y)<4] = P[X=1, Y=1] + P[X=1, Y=2] + P[X=2, Y=1]$$
$$= 0.1 + 0.2 + 0.1$$
$$= 0.4.$$

• **Example-2-** Let the joint probability mass function be defined by

$$f(x,y) = \frac{x+y}{21}, \quad x=1,2,3; \; y=1,2.$$

Then find the marginal probability mass functions of X and Y, and examine whether X and Y are independent?

• **Solution-** The marginal probability mass function of X is obtained as:

$$f_1(x) = \sum_y f(x,y) = \sum_{y=1}^{2} \frac{(x+y)}{21} = \frac{x+1}{21} + \frac{x+2}{21} = \frac{2x+3}{21}, \; x=1,2,3;$$

The marginal probability mass function of Y is obtained as:

$$f_2(y) = \sum_x f(x,y) = \sum_{x=1}^{3} \frac{(x+y)}{21} = \frac{1+y}{21} + \frac{2+y}{21} + \frac{3+y}{21} = \frac{6+3y}{21},$$

$y=1,2.$

Note that both $f_1(x)$ and $f_2(y)$ satisfy the properties of a probability mass function. Since

$$f(x,y) \neq f_1(x) f_2(y),$$

X and Y are dependent.

• <u>Example-3-</u> Let the joint probability mass function of X and Y be

$$f(x,y) = \frac{xy^2}{30}, \quad x=1,2,3;\ y=1,2.$$

Then find the marginal probability mass function of X and Y, and examine the independence of X and Y.

• <u>Solution-</u> The marginal probability mass function of X is obtained as:

$$f_1(x) = \sum_y f(x,y) = \sum_{y=1}^{2} \frac{xy^2}{30} = \frac{x}{30} + \frac{4x}{30} = \frac{5x}{30}$$

$$= \frac{x}{6}, \quad x=1,2,3$$

The marginal probability mass function of Y is given by

$$f_2(y) = \sum_x f(x,y) = \sum_{x=1}^{3} \frac{xy^2}{30} = \frac{y^2}{30} + \frac{2y^2}{30} + \frac{3y^2}{30}$$

$$= \frac{6y^2}{30} = \frac{y^2}{5}, \quad y=1,2.$$

We have

$$f_1(x) f_2(y)$$

$$= \frac{x}{6} \times \frac{y^2}{5} = \frac{xy^2}{30} = f(x,y) \text{ for } x=1,2,3 \text{ and } y=1,2.$$

Thus, $f(x,y) = f_1(x) f_2(y)$ for $x=1,2,3$ and $y=1,2$, and X and Y are independent.

• <u>Example-4-</u> Let the joint probability mass function of X and Y be

$$f(x,y) = \frac{xy^2}{13}, \quad (x,y) = (1,1), (1,2), (2,2)$$

Find marginal mass functions of X and Y. Also examine the independence of X and Y.

• Solution – The marginal probability mass function of X is:

$$f_1(x) = \begin{cases} \frac{5}{13}, & x=1 \\ \frac{8}{13}, & x=2 \end{cases}$$

and that of Y is:

$$f_2(y) = \begin{cases} \frac{1}{13}, & y=1 \\ \frac{12}{13}, & y=2. \end{cases}$$

Thus $f(x,y) \neq f_1(x) f_2(y)$ for $x=1,2$ and $y=1,2$, and X and Y are dependent.

Note that the support R of X and Y is "triangular". Whenever this support R is not "rectangular" the random variables must be dependent because R can not then equal the product set $\{(x,y): x \in R_1, y \in R_2\}$. That is, if we observe that the support R of X and Y is not a product set, then X and Y must be dependent. For illustration in example 4, X and Y are dependent because $R = \{(1,1), (1,2), (2,2)\}$ is not a product set.

• Example-5- Given the following bivariate probability distribution, obtain marginal distributions of X and Y, (ii) the conditional distribution of X given Y=2.

Y \ X	-1	0	1
0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
2	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$

• Solution – We have

Y \ X	-1	0	1	p(y) (Totals)
0	1/15	2/15	1/15	4/15
1	3/15	2/15	1/15	6/15
2	2/15	1/15	2/15	5/15
p(x) Totals	6/15	5/15	4/15	1

(i) Marginal distribution of X. From the above table, we get marginal distribution of X

X	−1	0	1
p(x)	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{4}{15}$

and Marginal distribution of Y :

Y	0	1	2
p(y)	$\frac{4}{15}$	$\frac{2}{5}$	$\frac{1}{3}$

(ii) Conditional distribution of X given Y=2, we have

$$P[X=x \cap Y=2] = P(Y=2) \cdot P(X=x|Y=2)$$

$$\Rightarrow P[X=x|Y=2] = \frac{P[X=x \cap Y=2]}{P[Y=2]}$$

$$\therefore P[X=-1|Y=2] = \frac{2/15}{1/3} = \frac{2}{5},$$

$$P[X=0|Y=2] = \frac{1/15}{1/3} = \frac{3}{15} = \frac{1}{5},$$

$$P[X=1|Y=2] = \frac{2/15}{1/3} = \frac{6}{15} = \frac{2}{5}.$$