## Construction of Confidence Intervals using pivots.

37/GC 379/MGB 375/GGD 4/R

### Pivotal Quantity Method

In many situations, the following simple method may be employed to obtain a confidence interval for a parametric function $\tau(\theta)$.

Suppose there exists a statistic $T$ and a function $\psi(T, \theta)$ of $T$ and $\theta$, which is measurable for each $\theta$, such that the distribution of $\psi(T, \theta)$ is independent of $\theta$ (i.e. the same for each $\theta$). Then $\psi(T, \theta)$ is called pivot and We can find two constants $k_1$ and $k_2$, depending on $\alpha_1$ and $\alpha_2$, but not on $\theta$, such that

$$P[\psi(T, \theta) < k_1] = \alpha_1, \quad P[\psi(T, \theta) > k_2] = \alpha_2.$$

so that $P[k_1 \leq \psi(T, \theta) \leq k_2] = 1 - \alpha \quad \forall \ \theta \in \Theta$, where $\alpha = \alpha_1 + \alpha_2$

Suppose further that it is possible to write the inequality

$$k_1 \leq \psi(T, \theta) \leq k_2.$$

in the form $\quad c_1(T) \leq \tau(\theta) \leq c_2(T)$.

Here $\psi(T, \theta)$ is so broken in two parts so that

$$c_1(T) \leq \tau(\theta) \leq c_2(T),$$

where $c_1(T)$ and $c_2(T)$ are independent of $\theta$. Then

$$P[c_1(T) \leq \tau(\theta) \leq c_2(T)] = P[k_1 \leq \psi(T, \theta) \leq k_2] = 1 - \alpha \quad \forall \ \theta \in \Theta.$$

Hence $c_1(T)$ and $c_2(T)$ may be taken as lower and upper confidence limits $T_1$ and $T_2$ respectively. In other words, for any given set of observations $x_1, x_2, \ldots, x_n$, the values $c_1(t)$ and $c_2(t)$ of $c_1(T)$ and $c_2(T)$ respectively are a pair of confidence limits to $\tau(\theta)$ with confidence coefficient $1 - \alpha$.



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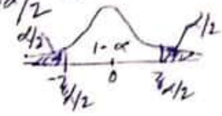
## Construction of Confidence Intervals using Pivots

<u>Example 1.</u> Let $x_1, x_2, \dots, x_n$ be a random sample from $N(\theta, \sigma^2)$, where $\sigma^2$ is supposed to be known. Obtain the confidence limits for $\theta$ having confidence coefficient $1-\alpha$.

<u>Soln:</u> In order to obtain the confidence limits for $\theta$, we may take a function of a statistic $T$ and $\theta$ such as

$$\psi(T,\theta) = \frac{\sqrt{n}\,(\bar{x}-\theta)}{\sigma} = Z, \text{ say} \qquad \bar{x} \sim N(\theta, \sigma^2/n) \Rightarrow$$

Now since $x_1, x_2, \dots, x_n$ are iid as $N(\theta, \sigma^2)$, $\psi(T,\theta) \sim N(0,1)$ which is independent of $\theta$. Further, we have $z_{\alpha/2}$ such that




$$P\left[-z_{\alpha/2} \le \sqrt{n}\,\frac{(\bar{x}-\theta)}{\sigma} \le z_{\alpha/2}\right] = 1-\alpha$$

or $$P\left[-\bar{x} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \le -\theta \le -\bar{x} + z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}\right] = 1-\alpha$$

or $$P\left[\bar{x} - z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}} \le \theta \le \bar{x} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right] = 1-\alpha$$

which has the probability $1-\alpha$ for every $\theta$.

therefore, $\bar{x} \mp \frac{\sigma}{\sqrt{n}} \cdot z_{\alpha/2}$ are the confidence limits for $\theta$ with confidence coefficient $1-\alpha$.

<u>Example 2:</u> Let $x_1, x_2, \dots, x_n$ be a random sample from $N(\mu, \sigma^2)$, where $\mu$ and $\sigma^2$ both are unknown and consider the problem of obtaining confidence interval for $\mu$.

<u>Soln:</u> Here, for $\theta = \mu$, and $\hat{\sigma}^2 = s^2$ we may consider

$$\psi(T,\theta) = \sqrt{n}\,\frac{(\bar{x}-\mu)}{s} \sim t_{n-1}$$

For every $\theta$ it has the $t$-distribution with $(n-1)$ d.f.. Hence the distribution of $\psi(T,\theta)$ is independent of $\theta$. Therefore,

$$P\left[-t_{\alpha/2, n-1} \le \frac{\sqrt{n}\,(\bar{x}-\mu)}{s} \le t_{\alpha/2, n-1}\right] = 1-\alpha$$

$$\Rightarrow P\left[\bar{x} - t_{\alpha/2,n-1}\cdot\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + t_{\alpha/2,n-1}\cdot\frac{s}{\sqrt{n}}\right] = 1-\alpha \quad \forall\, \theta$$

i.e. with confidence coefficient $1-\alpha$ the confidence limits are

$$\bar{x} \mp t_{\alpha/2, n-1}\cdot\frac{s}{\sqrt{n}}.$$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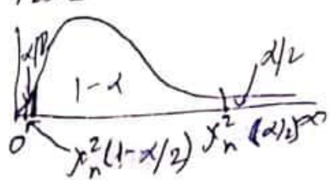
**Example 3:** Let $x_1, x_2, \ldots, x_n$ be a random sample from $N(\mu, \theta)$, where $\mu$ is known but $\theta$ is unknown. In order to obtain a confidence interval for $\theta$, we note that

$$\psi(T, \theta) = \sum_{i=1}^{n} (x_i - \mu)^2/\theta \sim \chi^2_n \quad \forall\ \theta$$

i.e. the distribution of $\psi(T, \theta)$ is independent of $\theta$.

Therefore, we have

$$P\left[\chi^2_n(1-\alpha/2) \le \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\theta} \le \chi^2_n(\alpha/2)\right] = 1-\alpha$$




or $$P\left[\sum_i (x_i-\mu)^2/\chi^2_n(\alpha/2) \le \theta \le \sum_i (x_i-\mu)^2/\chi^2_n(1-\alpha/2)\right] = 1-\alpha.$$

Thus a pair of confidence limits to $\theta$ with confidence coefficient $1-\alpha$ is $\sum_i (x_i-\mu)^2/\chi^2_n(\alpha/2)$ and $\sum_i (x_i-\mu)^2/\chi^2_n(1-\alpha/2)$.

**Example 4:** Suppose that $x_1, x_2, \ldots, x_n$ are a random sample from $N(\mu, \sigma^2)$, both being unknown. ~~and~~ Obtain the confidence limits for $\sigma^2 = \theta$.

**Soln:** Let $\psi(T, \theta) = \sum_{i=1}^{n} (x_i - \bar{x})^2/\sigma^2$, then

$$\psi(T, \theta) \sim \chi^2_{(n-1)} \quad \forall\ \theta \qquad \left\{\begin{array}{l}\text{1 d.f. is lost in order to estimate} \\ \mu \text{ by } \bar{x}.\end{array}\right.$$

i.e. the distribution of $\psi(T, \theta)$ is independent of $\theta$.

Proceeding as in the above example, we have the following pair of confidence limits to $\sigma^2 = \theta$, with confidence coefficient $1-\alpha$.

$$\sum_{i=1}^{n}(x_i-\bar{x})^2/\chi^2_{n-1}(\alpha/2) = (n-1)s^2/\chi^2_{n-1}(\alpha/2)$$

and $$\sum_{i=1}^{n}(x_i-\bar{x})^2/\chi^2_{n-1}(1-\alpha/2) = (n-1)s^2/\chi^2_{n-1}(1-\alpha/2),$$

where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$.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