# Probability Distribution

**Random variable**:

A variable whose value is determined by the outcome of a random experiment is called a *random variable*. Random variable is usually denoted by X.

A random variable may be discrete or continuous.

A *discrete random variable* is defined over a discrete sample space, i.e. the sample space whose elements are finite.

**Ex:** 1) Number of heads falling in a coin tossing
2) Number of points obtained when a die is thrown etc.

Discrete random variable takes values such as 0, 1, 2, 3, 4, …….

For example in a game of rolling of two dice there are certain outcomes.
Let X be a random variable defined as 'total points on two dice", then X takes values 2,3,4,5,6,7,8,9,10,11,12. Here X is a discrete variable.

A random variable which is defined over a continuous samplespace, i.e. the sample space whose elements are infinite is called a *continuous random variable*.

**Ex**: 1) All possible heights, weights, temperature, pressure etc.

Continuous random variable assumes the values in some interval (a,b) where a and b may be $\infty \quad and \quad -\infty$

**Probability distribution**:

The values taken by a discrete random variable such as X, and its associated probabilities P(X=x) or simply P(x) define a *discrete probability distribution* and P(X=x) is called "*Probability mass function*".

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The probability mass function (pmf) or probability function has the following *properties*.

1) $P(x) \geq 0$ , $\forall x$
2) $\sum_x P(X = x) = 1$

**Ex**: Suppose take a random experiment tossing of a coin two times. Then the sample space is S= {HH, HT, TH, TT}

Let X is a random variable denotes the 'number of heads' in the possible outcomes then X takes values 0, 1, 2. i.e. X=0, X=1 and X=2.

And their corresponding probabilities are

P(X=0) =1/4
P(X=1) =1/2
P(X=2) =1/4

So denoting possible values of X and by x and their probabilities by P(X=x), we have the following probability distribution.

| X | 0 | 1 | 2 |
|---|---|---|---|
| P(X=x) | 1/4 | 2/4 | 1/4 |

And P(X=0) + P(X=1) + P(X=2) = 1.

The values taken by a continuous random variable X, and its associated probabilities P(X=x) or simply P(x) define a *continuous probability distribution* and P(X=x) is called "*Probability density function*" and it has following properties

1) $P(x) \geq 0$, $\forall \; -\infty < x < \infty$

2) $\int_{-\infty}^{\infty} P(X = x) \; dx = 1$

**Mathematical Expectation**:

If X denotes a discrete random variable which can assume the values $x_1, x_2, x_3, \ldots\ldots x_k$ with respective probabilities $p_1, p_2, p_{3,\ldots\ldots\ldots\ldots,} p_k$ where $\sum_{i=1}^{k} p_i = 1$. The *Mathematical Expectation of X* denoted by E(X) is defined as

$$\boxed{E(X) = \sum_{i=1}^{k} p_i x_i}$$

**Ex**: **1)** A coin is tossed two times. Find the mathematical expectation of getting heads?
**Sol**: Define X as number of heads in tossing of two coins.
So X takes values 0, 1, 2.
And P(X=0) = 1/4
P(X=1) = 2/4
P(X=2) = 1/4

Now $E(X) = \sum_{i=1}^{k} p_i x_i$

$= 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 2 \times \frac{1}{4}$

$\Rightarrow$ E(X) =1.

**Ex: 2)** A die is thrown once. Find the mathematical expectation of the point on the upper face of the die.
**Sol:** Define a random variable X as the point on the upper face of the die.
So X takes values 1, 2, 3, 4, 5, 6.
The probability distribution is

| X | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| P(X=x) | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |

Now

$E(X) = \sum_{i=1}^{k} p_i x_i$

$= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$= \frac{21}{6} = 3.5$

**Def:** The mathematical expectation of g(x) is defined as

$$E[g(x)] = \sum_{i=1}^{k} g(x_i)\,p(x_i) \text{ provided } \sum_{i=1}^{k} |g(x_i)|\,p(x_i) \text{ is finite}$$

Putting $g(x) = x^2, x^3, 2x+4$

We get $E[x^2] = \sum_{i=1}^{k} x_i^{\,2} p(x_i)$

$E[x^3] = \sum_{i=1}^{k} x_i^{\,3} p(x_i)$

$E[2x+3] = \sum_{i=1}^{k} (2x_i+3)\,p(x_i)$

**Theorem:1)** If a and b are constants then $E[ax+b] = a\,E(x) + b$

**Proof:** $E[ax+b] = \sum_{i=1}^{k} (ax_i+b)\,p(x_i)$

$= \sum_{i=1}^{k} ax_i\,p(x_i) + \sum_{i=1}^{k} bp(x_i)$

$= a\,E(x) + b$

**Theorem:2)** If g(x) and h(x) are any two functions of a discrete random variable X, then

$$E[g(x) \pm h(x)] = E[g(x)] \pm E[h(x)]$$

**Proof:**

$$E[g(x) \pm h(x)] = \sum_{i=1}^{k} [g(x_i) \pm h(x_i)]\,p(x_i)$$

$$= \sum_{i=1}^{k} g(x_i)\,p(x_i) \pm \sum_{i=1}^{k} h(x_i)\,p(x_i)$$

$$= E[g(x)] \pm E[h(x)]$$

**Def**: The *'Variance'* of a random variable 'X' is given by

$$\boxed{\mu_2 = E[X - E(X)]^2}$$

**Theorem:3)** $E[X - E(X)]^2 = E[X^2] - [E(X)]^2$

**Proof:**

$$E[X - E(X)]^2 = \sum_{i=1}^{k} [x_i \pm E(X)]^2\,p(x_i)$$

$$= \sum_{i=1}^{k} x_i^{\,2} p(x_i) + \sum_{i=1}^{k} [E(X)]^2 p(x_i) - \sum_{i=1}^{k} 2x_i E(X)\,p(x_i)$$

$$= E[X^2] + [E(X)]^2 \sum_{i=1}^{k} p(x_i) - 2\,E[X] \sum_{i=1}^{k} x_i\,p(x_i)$$

$$= E[X^2] + [E(X)]^2 - 2\,E[X]\,E[X] \qquad [\because \sum_{i=1}^{k} p(x_i) = 1]$$

$$= E[X^2] - [E(X)]^2$$

**Theorem:4)** If X is a random variable and a and b are constants then $V[aX+b] = a^2\,V(X)$.

**Proof:**

$$
\begin{aligned}
V[aX+b] &= E[(aX+b)-E(aX+b)]^2 \\
&= E[(aX+b)-aE(X)-E(b)]^2 \\
&= E[(aX-aE(X)]^2 \\
&= E[a(X-E(X))]^2 \\
&= a^2\ E[(X-E(X)]^2 \\
&= a^2\,\mathrm{V(X)}.
\end{aligned}
$$

**Joint probability distribution**:

If X and Y are two discrete random variables, the probability for the simultaneous occurrence of X and Y can be represented by $P(X=x, Y=y)$ or $P(x, y)$, then a joint probability distribution is defined by all the possible values of X and Y associated with the joint probability density function $P(x, y)$. (X, Y) is said to be a two dimensional random variable.

In discrete case $P(x, y)$ have the following properties

i) $P(x, y) \geq 0 \;\; \forall \; x \, and \; y$

ii) $\sum_x \sum_y P(x, y) = 1$

**Ex:** If a coin is tossed two times

Define X as the result on the first coin i.e. H , T

Y as the result on the first coin i.e. H , T

Now the joint probability distribution of X and Y can be constructed as below

| Y \ X | H | T |
|---|---|---|
| H | 1/4 | 1/4 |
| T | 1/4 | 1/4 |

**Marginal density function**:

Given that the joint probability function $P(x, y)$ of the discrete random variables X and Y,

the *marginal density function of x* is defined as $P(x) = \sum_y P(x, y) \;\; \forall \; x$

the *marginal density function of y* is defined as $Q(y) = \sum_x P(x, y) \; \forall \; y$

**Conditional probability function**:

Given that the joint probability function $P(x, y)$ of the discrete random variables X and Y,

the Conditional probability function *of x given* y is defined as $P(x/y) = \dfrac{P(x, y)}{Q(y)}$ , $Q(y) > 0$

the Conditional probability function *of y given x* is defined as $P(y/x) = \dfrac{P(x, y)}{P(x)}$ , $P(x) > 0$

Where $Q(y)$ and $P(x)$ are the marginal density functions of y and x respectively.

**Independent random variables**:

Two discrete random variables X and Y are said to be *independent* if

$$P(x, y) = P(x)\ P(y)\ \forall\ x\, and\ y$$

**Note**: In case of independent random variables

$$P(x / y) = P(x)\ \forall\ x\, and\ y$$

and $P(y / x) = P(y)\ \forall\ x\, and\ y$

**Theorem**: If X and Y are two discrete random variables with joint probability function $P(x, y)$ then E [g(X) $\pm$ h(Y)] = E [g(X)] $\pm$ E [h(Y)].

**Proof**:

$$\text{E [g(x)} \pm \text{h(y)]} = \sum_i \sum_j [g(x_i) \pm h(y_i)]\ P(x_i, y_j)$$

$$= \sum_i \sum_j g(x_i)\ P(x_i, y_j) \pm \sum_i \sum_j h(y_i)\ P(x_i, y_j)$$

$$= \sum_i g(x_i)\ P(x_i) \pm \sum_j h(y_j)\ P(y_j) [\because P(x_i) = \sum_j P(x_i, y_j)$$

$$P(y_j) = \sum_i P(x_i, y_j)]$$

= E [g(X)] $\pm$ E [h(Y)]

Similarly putting g(X)=X and h(Y)=Y in the above theorem we get

E [X $\pm$ Y] = E [X] $\pm$ E [Y].

**Theorem**: If X and Y are two independent discrete random variables with joint probability function $P(x, y)$ then $E(XY) = E(X)E(Y)$

**Proof**:

$$E(XY) = \sum_i \sum_j x_i y_j\ P(x_i, y_j)$$

$$= \sum_i \sum_j x_i y_j\ P(x_i)P(y_i) \quad [\because \text{X and Y are independent}]$$

$$= \sum_i x_i\ P(x_i) \sum_j y_j\ P(y_j)$$

$$= E(X)\ E(Y)$$

**Covariance of X and Y**:

Covariance of X and Y is written symbolically as Cov(X, Y) or $\sigma_{XY}$ and is defined as

Cov(X, Y) = $E[X - E(X)][Y - E(Y)]$

And

Cov(X, Y) = $E[X - E(X)][Y - E(Y)]$

$$= E[XY - YE(X) - XE(Y) + E(X)E(Y)]$$

$$= E[XY] - E(Y)E(X) - E(X)E(Y) + E(X)E(Y)$$

$$= E[XY] - E(X)E(Y)$$

**Theorem**: If X and Y are independent then Cov(X, Y) = 0

**Proof**:

If X and Y are independent then we have

$$E(XY) = E(X)E(Y)$$
$$\Rightarrow E(XY) - E(X)E(Y) = 0$$
$$\Rightarrow \text{Cov}(X, Y) = 0$$

**Theorem**: If X and Y are two discrete random variables, then

$$V(aX + bY) = a^2V(X) + b^2V(Y) + 2abCov(X,Y)$$

**Proof**:

$$\begin{aligned} V(aX + bY) &= E[(aX + bY) - E(aX + bY)]^2 \\ &= E[(aX + bY) - aE(X) - bE(Y)]^2 \\ &= E[a\{X - E(X)\} + b\{Y - E(Y)\}]^2 \\ &= E[a^2\{X - E(X)\}^2 + b^2\{Y - E(Y)\}^2 + 2ab\{X - E(X)\}\{Y - E(Y)\}] \\ &= a^2E[X - E(X)]^2 + b^2E[Y - E(Y)]^2 + 2abE[X - E(X)][Y - E(Y)] \\ &= a^2V(X) + b^2V(Y) + 2abCov(X,Y) \end{aligned}$$

**Note**: If X and Y are independent then $V(aX + bY) = a^2V(X) + b^2V(Y)$ [$\because$ Cov(X, Y) = 0]

**Correlation coefficient**:

If $E(X^2)$ and $E(Y^2)$ exist, the correlation coefficient (is measure of relationship) between X and Y is defined as

$$\rho = \frac{Cov(X,Y)}{S.D(X)\,S.D(Y)}$$
$$= \frac{E[X - E(X)][Y - E(Y)]}{\sqrt{E[X - E(X)]^2}\,\sqrt{E[Y - E(Y)]^2}}$$

The sign of $\rho$ is determined by the sign of Cov(X, Y)

**Note**: If two variables X and Y are independent then $\rho$ =0 and X and Y are said to be uncorrelated.

**EXCERSICE**

1) Let X be a random variable having the following probability distribution.

| X | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| P(X=x) | 1/6 | 1/3 | 0 | 1/3 | 1/6 |

Find the expected values and the variance of i) X ii) 2X+1 iii) 2X-3

2) X is a randomly drawn number from the set {1,2,3,4,5,6,7,8,9,10}. Find E(2X-3).
3) X and Y are independent random variables with V(X) = V(Y) = 4. Find V(3X-2Y).
4) A bowl contains 6 chits, with numbers 1,2 and 3 written on two, three and one chits respectively. If X is the observed number on a randomly drawn chit, find $E(X^2)$.
5) Define independence of two random variables. For independent random variables X and Y, assuming the addition and multiplication law of expectation, prove that $V(aX+bY)=a^2V(X)+b^2V(Y)$
6) Let us consider the experiment of tossing an honest coin twice. The random variable X takes values 0 or 1 according as head or tail appears at the result of the first toss. The random variable Y takes values either 0 or 1 according as whether head or tail appears as a result of second toss. Show that X and Y are independent.
7) Ram and Shyam alternately toss a die, Ram starting the process. He who throws 5 or 6 first gets a prize of Rs.10 and the game ends with the award of the prize. Find the expectation of the Ram's gain.
8) Clearly explain the conditions under which, $E(XY)=E(X)E(Y)$ also prove the relation.
9) In a particular game a gambler can win a sum of Rs.100 with probability 2/5 and lose a sum of Rs.50 with probability 3/5. What is the mathematical expectation of his gain.
10) A business concern consists of 5 senior level and 3 junior level executives. A committee is to be formed by taking 3 executives at random. Find the expected number of senior executives to be in the committee.
11) There are two boxes: a white colored box and a red colored box. Each box contains 3 balls marked 1, 2, and 3. One ball is drawn from each box and their numbers are marked. Let X denotes the number observed from white box and Y denotes the number observed from the red box. Show that i) E(X+Y) = E(X) +E(Y)
    ii) E (XY) = E(X) E(Y)
    iii) V(X+Y) = V(X) + V(Y)
12) Calculate the mean and the standard deviation of the distribution of random digits, that is, f(X) = 1/10, X=0,1,2,…….,9
13) A person picks up 4 cards at random from a full deck. If he receives twice as many rupees as the number of aces he gets, find the expected gain.