Now consider the estimator

$$T = S_0^2 = \sum_{i=1}^{n} (X_i - \mu)^2 / n$$

Since $\sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\theta} \sim \chi^2_{(n)}$ and $E[\chi^2_{(n)}] = n$, $V(\chi^2_{(n)}) = 2n$

we have

$$E[S_0^2] = E[\theta \chi^2_{(n)}/n] = \frac{\theta}{n} E[\chi^2_{(n)}] = \frac{\theta}{n} . n = \theta.$$

and $$V[S_0^2] = V[\theta \chi^2_{(n)}/n] = \frac{\theta^2}{n^2} . 2n = 2\theta^2/n.$$

In other words, S_0^2 is an unbiased estimator and its variance coincides the Cramer-Rao lower bound for every θ. Hence, S_0^2 is a minimum variance unbiased estimator for $\theta = \sigma^2$.

Ex 1. Suppose $x_1, x_2, \dots, x_n$ are a random sample from $N(\mu, \sigma^2)$, σ^2 known and $\theta = \mu$. Consider the estimation of θ itself, the parameter space $\Theta = \{\theta : -\infty < \theta < \infty\}$.

Soln: In this case, the regularity conditions hold, provided that we consider proper estimators satisfying assumption iv). Now

$$f(x;\theta) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-(x-\theta)^2/2\sigma^2]$$

and therefore, $\log f(x;\theta) = c - (x-\theta)^2/2\sigma^2$,

where c is a constant independent of θ.

Now $\frac{\partial}{\partial\theta}\log f(x;\theta) = -\frac{2(x-\theta)}{2\sigma^2}(-1) = \frac{(x-\theta)}{\sigma^2}$

$$\Rightarrow E\left[\frac{\partial}{\partial\theta}\log f(x;\theta)\right]^2 = E\frac{[x-\theta]^2}{\sigma^4} = \frac{1}{\sigma^4}V(x) = \frac{\sigma^2}{\sigma^4} = \frac{1}{\sigma^2}$$

since $\tau'(\theta) = \frac{\partial}{\partial\theta}\theta = 1$, where $\tau(\theta) = \theta$.

the Cramer-Rao lower bound to the variance of an unbiased estimator of θ is

$$\frac{1}{n \cdot \frac{1}{\sigma^2}} = \sigma^2/n$$

Taking $T = \bar{x}$, the sample mean, the assumption iv) can be shown to be fulfilled. We know that $\bar{x} \sim N(\mu, \sigma^2/n)$. Thus $V(\bar{x}) = \sigma^2/n$ coincides with the Cramer-Rao lower bound for every θ. Hence, $\bar{x}$ is a minimum variance bound estimator of θ.

57 Example 2. Suppose again, that $X_1, X_2, \ldots, X_n$ are a random sample
GAD from $N(\mu, \sigma^2)$, the parameter μ being known. The unknown parameter is $\theta = \sigma^2$. Thus,

$$\Theta = \{\theta : 0 < \theta < \infty\}.$$

In this case too regularity conditions hold, provided that we take the right type of estimators, so that assumption iv) becomes valid.

Estimation of σ^2

Here, $\tau(\theta) = \theta$ and $\tau'(\theta) = 1$

Further, $f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-(x-\mu)^2/2\theta}$

$$\Rightarrow \log f(x,\theta) = c - \frac{1}{2}\log\theta - (x-\mu)^2/2\theta, \text{ where } c \text{ is independent of } \theta$$

Hence,

$$\frac{\partial}{\partial\theta}\log f(x;\theta) = -\frac{1}{2\theta} - \frac{(x-\mu)^2}{2\theta^2}(-1)$$

$$= \frac{(x-\mu)^2}{2\theta^2} - \frac{1}{2\theta}$$

and

$$\frac{\partial^2}{\partial\theta^2}\log f(x;\theta) = -\frac{(x-\mu)^2}{\theta^3} + \frac{1}{2\theta^2}$$

and accordingly,

$$E\left[\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\right] = -\frac{E(X-\mu)^2}{\theta^3} + \frac{1}{2\theta^2}$$

$$= -\frac{V(X)}{\theta^3} + \frac{1}{2\theta^2} = -\frac{\theta}{\theta^3} + \frac{1}{2\theta^2} = -\frac{1}{\theta^2} + \frac{1}{2\theta^2} = -\frac{1}{2\theta^2}$$

Using alternative form of CR inequality for particular case

$$V(T) \geq \frac{-[\tau'(\theta)]^2}{nE\left[\frac{\partial^2}{\partial\theta^2}\log f(x,\theta)\right]}$$

We find that, the Cramer-Rao lower bound for the variance of an Unbiased estimator of θ is $-\frac{1}{n(-\frac{1}{2\theta^2})} = 2\theta^2/n$.

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CBCS Course

S.S. in Statistics
Vikram University, Ujjain

MA/MSc. Statistics, Second Semester

Paper-I : Statistical Inference - I.

Unit-II: Lecture-3 ,

Title : Examples on Cramer-Rao Inequality

Teacher: Dr. Praveen Gupta
MblNo. : 94071-58286