The Cramer-Rao inequality provides, only the method of deciding whether or not a given estimator is a Minimum Variance Bound estimator. Also, this method assumes very stringent regularity conditions. One more thing is that an estimator which is not MVB, does not imply that it is not MVU estimator. To remove these defects and to obtain an MVU estimator C.R. Rao and D. Blackwell have given the following theorem.

<u>Theorem</u>: (Rao-Blackwell Theorem)

Let the statistic $U = U(X_1, X_2, \ldots, X_n)$ be an unbiased estimator of $\tau(\theta)$, while T is a sufficient statistic for θ. Consider the function $\phi(T)$ of T such that

$$\phi(T) = E[U|T]$$

Then $\phi(T)$ is itself an unbiased estimator of $\tau(\theta)$ and

$$V[\phi(T)] \leq V[U].$$

<u>Proof</u>: Here, we note that T is sufficient for θ, so $E[U|T]$ is independent of θ.

Now
$$E[\phi(T)] = E[E(U|T)] = \int_{-\infty}^{\infty} E(U|T)\, f_T(t)\, dt$$
$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} u \cdot f_{U|T}(u|t)\, du\right] f_T(t)\, dt$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\,[f_{U|T}(u|t) \cdot f_T(t)]\, du\, dt$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \cdot f_{U,T}(u,t)\, du\, dt$$
$$= E[U] = \tau(\theta) \quad \ldots (1), \text{ since } U \text{ is unbiased for } \tau(\theta)$$

Again,
$$E[V(U|T)] = E[E(U^2|T)] - E[E(U|T)]^2$$
$$= E[U^2] - \{E[U]\}^2 - E[E(U|T)]^2 + \{E[U]\}^2$$

since $E[E(U|T)] = E(U)$ in eq. (1) $\Rightarrow E[E(U^2|T)] = E[U^2]$

$$= V[U] - E[\phi(T)]^2 + \{E[\phi(T)]\}^2 \quad \text{since } E[U] = E[\phi(T)] = \tau(\theta)$$
$$= V(U) - V[\phi(T)] \geq 0, \text{ since } E[V(U|T)] \geq 0.$$

$\Rightarrow V[U] \geq V[\phi(T)] \Rightarrow V[\phi(T)] \leq V[U]$ ...

Note: Unless $U = \phi(T)$ with probability measure one,

$$E[V(U|T)] = E[U - \phi(T)]^2 > 0$$

and we have strict inequality $V(U) > V[\phi(T)]$.

Thus, we have seen that a given unbiased estimator U can be improved upon by forming the new estimator $\phi(T)$ based on U and sufficient statistic T. This process of finding an improved estimator (with respect to unbiasedness and minimum variance), starting from an unbiased estimator is called Blackwellisation after D. Blackwell.

The estimator $\phi(T)$ will not only be a better estimator than U, but also be the best when T has a complete family of distributions.

CBCS Course

S. S. in Statistics

Vikram University, Ujjain

Class : MA/MSc. Second Semester

Subject : Statistics

Paper : First, Statistical Inference-I

Unit : Second

Lecture : No. 5

Title : Rao-Blackwell Theorem and Blackwellization

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